Exercise 2C

1 a $a^2 = 9 \quad b^2 = 5$

$b^2 = a^2(1-e^2) \Rightarrow \frac{5}{9} = 1-e^2$

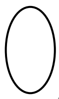
$\Rightarrow e^2 = \frac{4}{9}$ so $e = \frac{2}{3}$

b $a^2 = 16 \quad b^2 = 9$

$b^2 = a^2(1-e^2) \Rightarrow \frac{9}{16} = 1-e^2$

$\Rightarrow e^2 = \frac{7}{16}$ so $e = \frac{\sqrt{7}}{4}$

c $a^2 = 4 \quad b^2 = 8$

Need to use $a^2 = b^2(1-e^2)$ since $b > a$, so the ellipse is  shape.

$\frac{4}{8} = 1-e^2 \Rightarrow e^2 = \frac{1}{2}$ so $e = \frac{1}{\sqrt{2}}$

2 a $a^2 = 4 \quad b^2 = 3$

$b^2 = a^2(1-e^2) \Rightarrow \frac{3}{4} = 1-e^2$

$\Rightarrow e^2 = \frac{1}{4}$ so $e = \frac{1}{2}$

Foci are at $(\pm ae, 0) = (\pm 1, 0)$

Directrices are $x = \pm \frac{a}{e} \Rightarrow x = \pm 4$

b $a^2 = 16 \quad b^2 = 7$

$b^2 = a^2(1-e^2) \Rightarrow \frac{7}{16} = 1-e^2$

$\Rightarrow e^2 = \frac{9}{16}$ so $e = \frac{3}{4}$

Foci are at $(\pm ae, 0) = (\pm 3, 0)$

Directrices are $x = \pm \frac{a}{e} \Rightarrow x = \pm \frac{16}{3}$

2 c $a^2 = 5, b^2 = 9$

Since $b > a$, use

$a^2 = b^2(1-e^2) \Rightarrow \frac{5}{9} = 1-e^2$

$\Rightarrow e^2 = \frac{4}{9}$ so $e = \frac{2}{3}$

Foci are at $(0, \pm be)$, i.e. foci are $(0, \pm 2)$

Directrices are $y = \pm \frac{b}{e}$, i.e. $y = \pm \frac{9}{2}$

3 a Since the focus is on the x-axis and the directrix is parallel to the y-axis, we know that $a > b$. In fact, the major axis of the ellipse, on which the foci lie, has to be perpendicular to the directrix, so knowing that one focus is on the x-axis and that the directrix is perpendicular to it is enough to identify the major axis of the ellipse as lying on the x-axis.

b i The directrix is at $x = 12$, so $\frac{a}{e} = 12$

$\Rightarrow a = 12e$

The focus is at $(ae, 0)$, so $ae = 3$

$12e \times e = 3 \Rightarrow e^2 = \frac{1}{4}$ so $e = \frac{1}{2}$

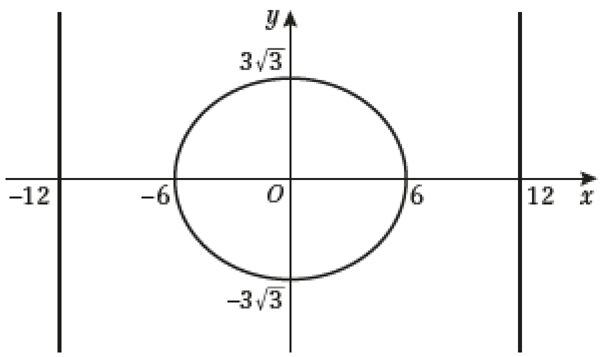
ii Since $ae = 3$, $a = 6$

Using $b^2 = a^2(1-e^2)$

$b^2 = 36(1-\frac{1}{4}) = 36 \times \frac{3}{4} = 27$

$\Rightarrow b = 3\sqrt{3}$

c



4 a Since the directrix is parallel to the x-axis, and the focus is on the y-axis, we know that $b > a$.

b i The directrix is at $y = 8$, so $\frac{b}{e} = 8$

$\Rightarrow b = 8e$

The focus is at $(0, be)$, so $be = 2$

$8e \times e = 2 \Rightarrow e^2 = \frac{1}{4}$ so $e = \frac{1}{2}$

ii Since $be = 2$, $b = 4$

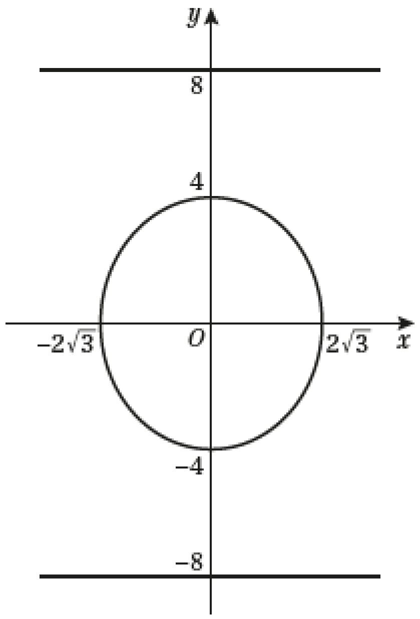
As $b > a$, use

$a^2 = b^2(1 - e^2)$

$a^2 = 16(1 - \frac{1}{4}) = 16 \times \frac{3}{4} = 12$

$\Rightarrow a = 2\sqrt{3}$

c



5 a $\frac{x^2}{5} - \frac{y^2}{3} = 1 \Rightarrow a^2 = 5, b^2 = 3$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{3}{5} = e^2 - 1$

$\Rightarrow e^2 = \frac{8}{5}$ so $e = \frac{2\sqrt{2}}{\sqrt{5}} = \frac{2\sqrt{10}}{5}$

b $\frac{x^2}{9} - \frac{y^2}{7} = 1 \Rightarrow a^2 = 9, b^2 = 7$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{7}{9} = e^2 - 1$

$\Rightarrow e^2 = \frac{16}{9}$ so $e = \frac{4}{3}$

5 c $\frac{x^2}{9} - \frac{y^2}{16} = 1 \Rightarrow a^2 = 9, b^2 = 16$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{16}{9} = e^2 - 1$

$\Rightarrow e^2 = \frac{25}{9}$ so $e = \frac{5}{3}$

6 a $\frac{x^2}{4} - \frac{y^2}{8} = 1$

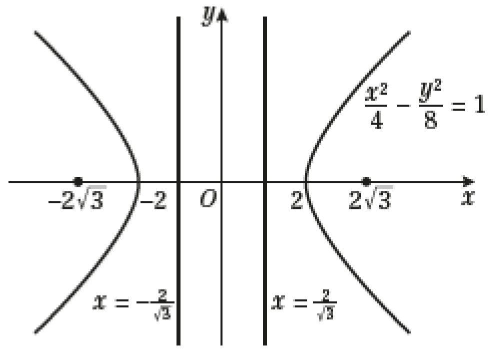
$a = 2, b = 2\sqrt{2}$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{8}{4} = e^2 - 1$

$\Rightarrow e = \sqrt{3}$

Foci are at $(\pm 2\sqrt{3}, 0)$

Directrices are $x = \pm\frac{2}{\sqrt{3}}$



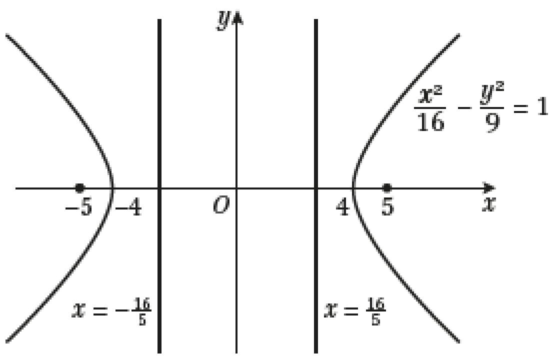
b $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$a = 4, b = 3$

$\Rightarrow 9 = 16(e^2 - 1) \Rightarrow e^2 = \frac{25}{16} \Rightarrow e = \frac{5}{4}$

Foci are at $(\pm 5, 0)$

Directrices are $x = \pm\frac{16}{5}$



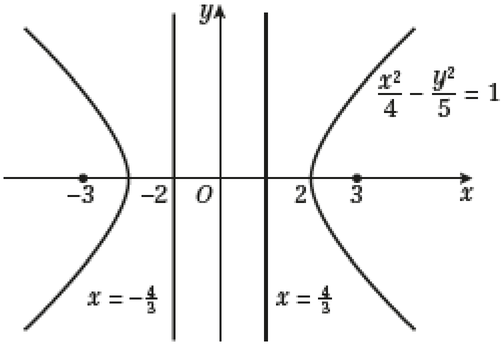
6 c $\frac{x^2}{4}-\frac{y^2}{5}=1$

$a=2,\ b=\sqrt{5}$

$\Rightarrow 5=4\left(e^2-1\right)\Rightarrow e^2=\frac{9}{4}\Rightarrow e=\frac{3}{2}$

Foci are at (±3, 0)

Directrices are $x=\pm\frac{4}{3}$



7 a The eccentricity, e, of a hyperbola is given by $b^2=a^2\left(e^2-1\right)$

Rearranging, $e=\sqrt{1+\frac{b^2}{a^2}}$

and the foci have coordinates $(\pm ae, 0)$
Then we just need to compute the eccentricity in each case, as follows:

i $e=\sqrt{1+\frac{1}{24}}=\frac{5}{\sqrt{24}}$;

then $ae=\frac{5}{\sqrt{24}}\times\sqrt{24}=5$

ii $e=\sqrt{1+24}=5$;
then $ae=5\times1=5$

iii $e=\sqrt{1+\frac{9}{16}}=\frac{5}{4}$;

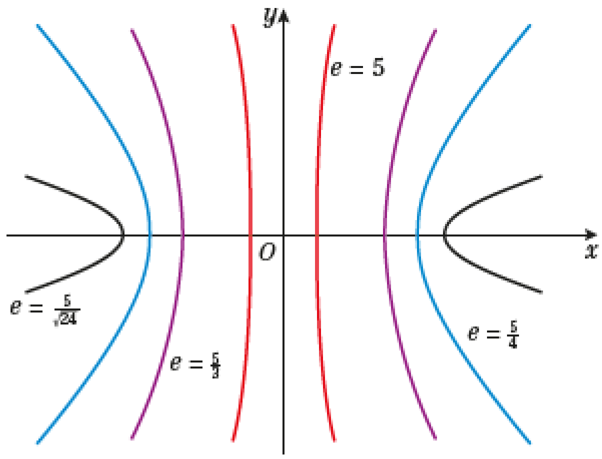
then $ae=\frac{5}{4}\times4=5$

iv $e=\sqrt{1+\frac{16}{9}}=\frac{5}{3}$;

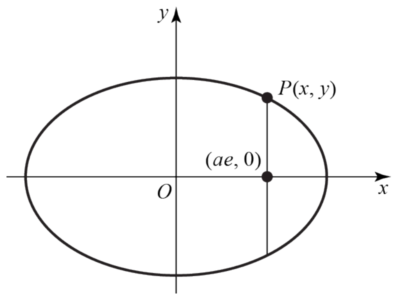
then $ae=\frac{5}{3}\times3=5$

Since for all four hyperbolas $ae=5$, all have foci at (±5, 0)

7 b We have already found the values of the eccentricity: $\frac{5}{\sqrt{24}}$, 5, $\frac{5}{4}$ and $\frac{5}{3}$



8 Since $a>b$, the ellipse has its major axis along the x-axis. Let P be a point of intersection of the chord with the ellipse, with coordinates (x, y).



The focus $(ae, 0)$ is on the chord, so $x=ae$.
Substitute into the equation for the ellipse:

$\frac{(ae)^2}{a^2}+\frac{y^2}{b^2}=1$

$\Rightarrow y^2=b^2\left(1-e^2\right)$

From the definition of eccentricity,

$b^2=a^2\left(1-e^2\right)$ so $e^2=1-\frac{b^2}{a^2}$

Substituting for e^2 in the equation for y,

$y^2=b^2\left(1-1+\frac{b^2}{a^2}\right)\Rightarrow y=\pm\frac{b^2}{a}$

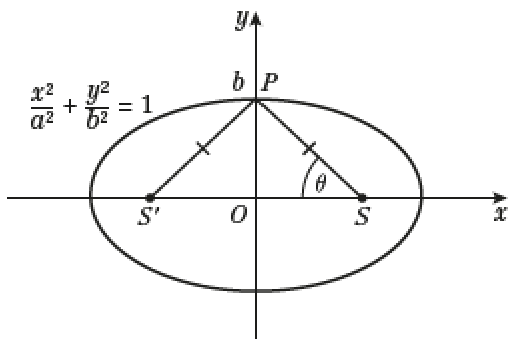
The length of the latus rectum is $2y=\frac{2b^2}{a}$

9 a Assume the foci are on the x-axis. The distance between the foci is $2ae = 16$, so $ae = 8$
The distance between the directrices is
$$\frac{2a}{e} = 25$$
Substituting for a gives
$$25e^2 = 16 \Rightarrow e = \frac{4}{5}$$

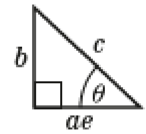
b The foci are on the y-axis, thus
$$b = \frac{8}{e} = 8 \times \frac{5}{4} = 10$$
and $a = \left(\sqrt{1 - \frac{16}{25}}\right)10 = \frac{3}{5} \times 10 = 6$
Then the equation of the ellipse is
$$\frac{x^2}{36} + \frac{y^2}{100} = 1$$

10 Rewrite the equation of the ellipse by dividing both sides by 36
$\frac{x^2}{36} + \frac{y^2}{9} = 1$ so $a = 6$ and $b = 3$
Then the eccentricity of the ellipse is
$$e = \sqrt{1 - \frac{9}{36}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$
The points A and B have coordinates $(\pm ae, 0)$, so they are the foci of the ellipse.
Using the focus and directrix definitions of an ellipse, for any point P with coordinates (x, y),
$$PA + PB = e\left(\frac{a}{e} + x\right) + e\left(\frac{a}{e} - x\right)$$
$$= 2a = 12$$

11



Consider ΔPOS

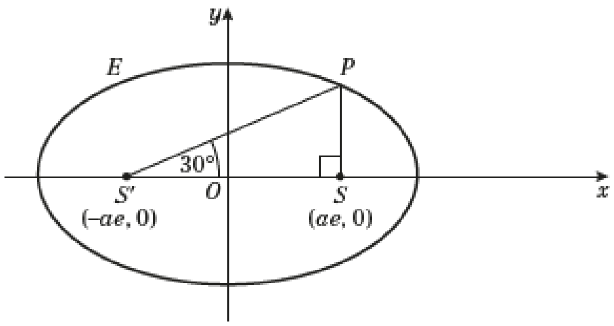


$c^2 = b^2 + a^2e^2$, but $b^2 = a^2(1 - e^2)$
$\Rightarrow c^2 = a^2 - a^2e^2 + a^2e^2 = a^2$
$\Rightarrow c = a$

So $\cos\theta = \frac{ae}{a} = e$

> If you use the result that $SP + S'P = 2a$ then since $S'P = SP$ it is clear $SP = a$
>
> Hence $\cos\theta = \frac{ae}{a} = e$

12



PS is y where $\dfrac{a^2e^2}{a^2}+\dfrac{y^2}{b^2}=1$

$$y^2=b^2\left(1-e^2\right)$$

$$y=b\sqrt{1-e^2}$$

$$SS'=2ae$$

$$\tan 30°=\frac{1}{\sqrt{3}}=\frac{y}{2ae}=\frac{b\sqrt{1-e^2}}{2ae}$$

But $b^2=a^2\left(1-e^2\right)$

$$\Rightarrow \frac{1}{\sqrt{3}}=\frac{a\sqrt{1-e^2}\sqrt{1-e^2}}{2ae}$$

$$\Rightarrow \frac{2e}{\sqrt{3}}=1-e^2$$

$$\Rightarrow e^2+\frac{2}{\sqrt{3}}e-1=0$$

Completing the square,

$$e^2+\frac{2}{\sqrt{3}}e+\frac{1}{3}=1+\frac{1}{3}$$

$$\Rightarrow \left(e+\frac{1}{\sqrt{3}}\right)^2=1+\frac{1}{3}=\frac{4}{3}$$

$$\Rightarrow e+\frac{1}{\sqrt{3}}=\frac{2}{\sqrt{3}} \text{ so } e=\frac{1}{\sqrt{3}} \quad (e>0)$$